RANK GRADIENT OF SMALL COVERS

DARLAN GIRÃO

ABSTRACT. We prove that if $M \longrightarrow P$ is a small cover of a compact right-angled hyperbolic polyhedron then M admits a cofinal tower of finite sheeted covers with positive rank gradient. As a corollary, if $\pi_1(M)$ is commensurable with the reflection group of P, then M admits a cofinal tower of finite sheeted covers with positive rank gradient.

1. Introduction

Let P^n be an n-dimensional *simple convex polytope*. Here P^n is simple if the number of codimension-one faces meeting at each vertex is n. Equivalently, the dual K_P of its boundary complex ∂P^n is an $(n-1)$-dimensional simplicial sphere. A *Small Cover* of P^n is an n-dimensional manifold endowed with an action of the group $\mathbb{Z}_2^n$ whose orbit space is P^n. The notion of small cover was introduced and studied by Davis and Januszkiewicz in [DJ]. We will be dealing mostly with 3-dimensional polytopes. In the case P is a compact right-angled polyhedron in $\mathbb{H}^3$ then Andreev's theorem [An] implies that all vertices have valence three and in particular P is a simple convex polytope.

Let G be a finitely generated group. The *rank of* G is the minimal number of elements needed to generate G, and is denoted by $\mathrm{rk}(G)$. If G_j is a finite index subgroup of G, the Reidemeister-Schreier process [LS] gives an upper bound on the rank of G_j.

$$\mathrm{rk}(G_j) - 1 \leq [G : G_j](\mathrm{rk}(G) - 1)$$

Recently Lackenby introduced the notion of *rank gradient* [La]. Given a finitely generated group G and a collection $\{G_j\}$ of finite index subgroups, the *rank gradient* of the pair $(G, \{G_j\})$ is defined by

$$\mathrm{rgr}(G, \{G_j\}) = \lim_{j\to\infty} \frac{\mathrm{rk}(G_j) - 1}{[G : G_j]}$$

We say that the collection of finite index subgroups $\{G_j\}$ is *cofinal* if $\cap_j G_j = \{1\}$, and we call it a *tower* if $G_{j+1} < G_j$.

In general it is very hard to construct co-final families $(G, \{G_j\})$ with positive rank gradient. For instance, it seems that only recently the

first examples of torsion-free finite covolume Kleinian groups with this property were given in [Gi]. Before stating main result we need some terminology.

If M is a finite volume hyperbolic 3-manifold, we call the family of covers $\{M_j \longrightarrow M\}$ *cofinal* (resp. a *tower*) if $\{\pi_1(M_j)\}$ is cofinal (resp. a tower). By rank gradient of the the pair $(M, \{M_j\})$, $\mathrm{rgr}(M, \{M_j\})$, we mean the rank gradient of $(\pi_1(M), \{\pi_1(M_j)\})$.

Theorem 1.1. *Let $M \longrightarrow P$ be a small cover of a compact, right-angled hyperbolic polyhedron of dimension 3. Then M admits a cofinal tower of finite sheeted covers $\{M_j \longrightarrow M\}$ with positive rank gradient.*

We remark here that this is not true for 3-dimensional polytopes in general. Let $T^3 \longrightarrow C$ be the covering of a cube in 3-dimensional Euclidean space by the 3-torus T^3. It is easy to see that any subgroup of $\pi_1(T^3) = \mathbb{Z}^3$ has bounded rank and therefore rank gradient with respect to any tower of covers is zero.

This theorem has the following consequence

Corollary 1.2. *Let M be a finite volume hyperbolic 3-manifold such that $\pi_1(M)$ is commensurable with the group generated by reflections along the faces of a compact, right-angled hyperbolic polyhedron $P \subset \mathbb{H}^3$. Then M admits a cofinal tower of finite sheeted covers $\{M_j \longrightarrow M\}$ with positive rank gradient.*

We note that this corollary is complementary to the results of [Gi], where ideal right-angled polyhedra were considered. The key idea there was to estimate the rank of the fundamental group of the manifolds by estimating their number of cusps. Here the estimates on the rank of the fundamental groups are given in terms of the rank of the $\mod 2$ homology.

The study of the rank of the fundamental group of (finite volume hyperbolic) 3-manifolds has always been a central theme in low dimensional topology. In recent years the study of the rank gradient for this class of groups has received special attention. For instance, motivated by the seminal paper [La1] of Lackenby, Long–Lubotzky–Reid [LLR] prove that every finite volume hyperbolic 3-manifold has a cofinal tower of covers in which the *Heegaard genus* grows linearly with the degree of the covers. Whether or not the same happens to the rank of their fundamental groups is a major open problem. Another important recent work using these notions is [AN]. There they connect the problem related to the growth of the rank of π_1 and the growth of the Heegaard genus in a cofinal tower of hyperbolic 3-manifolds to a

problem in topological dynamics, *the fixed price problem* (see [Fa] and [Ga]). These papers have all been motivation for the current work.

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2. Small covers

Recall that an n-dimensional convex polytope P^n is simple if the number of codimension-one faces meeting at each vertex is n. Equivalently, the dual K_P of its boundary complex ∂P is an $(n-1)$-dimensional simplicial complex. A Small Cover of P is an n-dimensional manifold endowed with an action of the group $\mathbb{Z}_2^n$ whose orbit space is P.

Let K be a finite simplicial complex of dimension $n-1$. For $0 \leq i \leq n-1$, let f_i be the number of i-simplices of K. Define a polynomial $\Phi_K(t)$ of degree n by

$$\Phi_K(t) = (t-1)^n + \sum_{i=0}^{n-1} f_i (t-1)^{n-1-i}$$

and let h_i be the coefficient of t^{n-i} in this polynomial, i.e.,

$$\Phi_K(t) = \sum_{i=0}^{n} h_i t^{n-i}$$

If we restrict to the case where K is the dual K_P of the boundary complex of a convex simple polytope P^n, then one can see that f_i is the number of faces of P^n of codimension $i+1$. Let $h_i(P^n)$ denote the coefficient of t^{n-i} in $\Phi_{K_P}(t)$

One of the main results of [DJ], here stated in a very particular setting, is

Theorem 2.1. *Let $\pi : M^n \longrightarrow P^n$ be a small cover of a simple convex polytope P^n and let $b_i(M^n, \mathbb{Z}_2)$ be the i^{th} $\bmod 2$ Betti number of M^n. Then $b_i(M^n, \mathbb{Z}_2) = h_i(P^n)$*

As observed in [DJ], it is somewhat surprising that all $\bmod 2$ Betti numbers of a small cover M^n depend on P^n only. They also show that this theorem does not hold for homology groups in general. They

provide small covers of a square Q by tori and a Klein bottles are such that the rational Betti numbers are not determined by Q.

When P is a right-angled dodecahedron in $\mathbb{H}^3$ then [GS] shows that up to homeomorphism there exists exactly 25 small covers of P. [Ch] estimates the number of orientable small covers of the n-dimensional cube. Also, if P is a 3-dimensional convex polytope, [NN] proves that P admits an orientable small cover. They also prove that unless P is a 3-simplex, then it admits a non-orientable small cover.

3. Proof of theorem

In this section we prove

Theorem 1.1. *Let $M \longrightarrow P$ be a small cover of a compact, right-angled hyperbolic polyhedron of dimension 3. Then M admits a cofinal tower of finite sheeted covers $\{M_j \longrightarrow M\}$ with positive rank gradient.*

Proof. As observed above, when P be a compact right-angled polyhedron in $\mathbb{H}^3$ then Andreev's theorem ([An]) implies that all vertices have valence three and in particular P is a simple convex polytope. Let V, E and F denote the number of vertices, edges and faces, respectively, of a 3-dimensional simple polyhedron P. Straightforward computations show that

$$\Phi_{K_P}(t) = t^3 + (F-3)t^2 + (3-2F+E)t + (V-E+F-1)$$

and thus $h_0(P) = 1$, $h_1(P) = F-3$, $h_2(P) = 3-2F+E$ and $h_3(P) = V-E+F-1$. Since P is simple we also have $E = 3V/2$. And since $V-E+F = 2$ (∂P is topologically a sphere) this gives $F = (1/2)V+2$ and therefore $h_1(P) = (1/2)V-1$.

The strategy involved in the proof is similar to the proof of the main theorem in [Gi]. Given $P \in \mathbb{H}^3$, construct a family of polyhedra

$$P = P_0, P_1, \ldots, P_j, \ldots$$

such that P_{j+1} is obtained from P_j by reflecting P_j along one of its faces. This must be done in a way such that the following holds: if $x \in \mathbb{H}^3$, then there exists j sufficiently large so that x lies in the interior of P_j. This means that the family $\{P_j\}$ is an exhaustion of $\mathbb{H}^3$. Denote by G_j the group generated by reflections along the faces of P_j. If the family $\{P_j\}$ is constructed as above, then it is easy to see that $G_{j+1} < G_j$ (with index 2) and it can be shown that the tower $\{G_j\}$ is cofinal (see [Ag]). We refer the reader to [Gi] for a detailed proof of this fact.

Now let $M \longrightarrow P$ be a small cover of P, and let $M_j \longrightarrow M$ be the cover corresponding to the group $\pi_1(M) \cap G_j$. Recall that the degree of the cover $M \longrightarrow P$ is 2^3.

Lemma 3.1. $[\pi_1(M_j) : \pi_1(M_{j+1})] = 2$

Proof of lemma. First observe that $[G_j : G_{j+1}] = 2$. Since $\pi_1(M_1) = G_1 \cap \pi_1(M)$, we must have $[\pi_1(M) : \pi_1(M_1)] \leq 2$. If this index were 1, then it would mean that $\pi_1(M_1) = \pi_1(M) < G_1$ from which would follow that M_1 is a manifold cover of the simple polyhedron P_1 of degree 2^2. But this is not possible, since any manifold cover of a 3-dimensional simple polyhedron must have degree at least 2^3 (see [DJ], [GS]). The remaining cases follow by induction. □

Since $[G_j : G_{j+1}] = 2$, from the above lemma and an inductive argument we see that $M_j \longrightarrow P_j$ is a cover of degree 2^3. In particular this implies that M_j is a small cover of P_j. From theorem 2.1 we have

$$b_1(M_j, \mathbb{Z}_2) = h_1(P_j)$$

Denote by V_j the number of vertices of P_j. From the computations of h_1,

$$b_1(M_j, \mathbb{Z}_2) = h_1(P_j) = \frac{V_j}{2} - 1$$

Also note that a lower bound for $\mathrm{rk}(\pi_1(M_j))$ is $b_1(M_j, \mathbb{Z}_2)$ and thus

$$\mathrm{rk}(\pi_1(M_j)) \geq \frac{V_j}{2} - 1$$

We also have $[\pi_1(M) : \pi_1(M_j)] = 2^j$. Therefore

$$\mathrm{rgr}(\pi_1(M)), \{\pi_1(M_j)\}) = \lim_{j\to\infty} \frac{\mathrm{rk}(\pi_1(M_j)) - 1}{[\pi_1(M) : \pi_1(M_j)]} \geq \lim_{j\to\infty} \frac{V_j - 3}{2^{j+1}}$$

We thus need to show that V_j is of magnitude 2^j. This is a consequence of a theorem of Atkinson [At].

Theorem 3.2 (Atkinson)**.** *There exist constants $C, D > 0$ such that if P is a compact right-angled polyhedron in $\mathbb{H}^3$ with V vertices then*

$$C(V - 8) \leq vol(P) \leq D(V - 10)$$

We now observe that, in our setting, $\mathrm{vol}(P_j) = 2^j \mathrm{vol}(P)$ and thus

$$D(V_j - 10) \geq 2^j \mathrm{vol}(P) \geq 2^j C(V - 8)$$

which gives

$$V_j \geq 2^j \frac{C}{D}(V - 8) + 10$$

where V is the number of vertices in P. Also, the second inequality in Atkinson's theorem provide $V > 8$. The theorem is now proved. □

4. Extending the examples

Theorem 1.1 has an interesting corollary, which complements the family of manifolds provided in [Gi].

Corollary 1.2. *Let N be a closed hyperbolic 3-manifold such that $\pi_1(N)$ is commensurable with the group generated by reflections along the faces of a compact, right-angled hyperbolic polyhedron $P \subset \mathbb{H}^3$. Then N admits a cofinal tower of finite sheeted covers $\{N_j \longrightarrow N\}$ with positive rank gradient.*

Proof. First we note that, by passing to a finite cover, we may assume N is orientable. Note also that [NN] implies orientable small covers of P exist and therefore N is commensurable with a small cover $M \longrightarrow P$. Let N' be the manifold cover of both M and N corresponding to the group $\pi_1(M) \cap \pi_1(N)$. Consider now $N_j \longrightarrow N$ corresponding to the group $\pi_1(N') \cap G_j$, where the family $\{G_j\}$ is given as in the proof of theorem 1.1. Consider also $\{M_j\}$ the tower where M_j is a small cover of P_j, also as in the proof of theorem 1.1.

Note that $\pi_1(N_j) = \pi_1(N') \cap G_j = \pi_1(N') \cap \pi_1(M_j) < \pi_1(M_j)$ and therefore we have the following diagram of covers, where the labels in the arrows indicate the degree of the cover.

$$\begin{array}{ccccccccccc} & & P & \overset{2}{\leftarrow} & P_1 & \overset{2}{\leftarrow} & \cdots & \overset{2}{\leftarrow} & P_j & \overset{2}{\leftarrow} & \cdots \\ & & \uparrow 2^3 & & \uparrow 2^3 & & & & \uparrow 2^3 & & \\ & & M & \overset{2}{\leftarrow} & M_1 & \overset{2}{\leftarrow} & \cdots & \overset{2}{\leftarrow} & M_j & \overset{2}{\leftarrow} & \cdots \\ & & \uparrow & & \uparrow & & & & \uparrow & & \\ N & \leftarrow & N' & \leftarrow & N_1 & \leftarrow & \cdots & \leftarrow & N_j & \leftarrow & \cdots \end{array}$$

Agol-Culler-Shallen proved the following in [ACS] (see also [Sh]).

Theorem 4.1. *Let M be a closed, orientable hyperbolic 3 manifold such that $b_1(M, \mathbb{Z}_p) = r$ for a given prime p. Then for any finite sheeted covering space M' of M, $b_1(M', \mathbb{Z}_p) \geq r - 1$.*

We thus have

$$\mathrm{rk}(\pi_1(N_j)) \geq b_1(N_j, \mathbb{Z}_2) \geq b_1(M_j, \mathbb{Z}_2) - 1 = \frac{V_j}{2} - 2$$

and therefore all we need to do is show that $[\pi_1(N) : \pi_1(N_j)]$ grows at most as fast as 2^j. But from the above diagram we see that $[\pi_1(N_j) : \pi_1(N_{j+1})] \leq 2$ and we are done. □

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Department of Mathematics,
Universidade Federal do Ceará
E-mail: dgirao@mat.ufc.br